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INVESTIGATIONS INTO MATHEMATICAL
EFFECTIVENESS CRITERIA

Timothy J. Horrigan

COOK ELECTRIC COMPANY
Cook Technological Center Division
Morton Grove, Illinois

Contract No. AF19(604)-7497

Project No. 5632

Task No. 563203

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ELECTRONICS RESEARCH DIRECTORATE
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ABSTRACT

The objective of this investigation is the formulation of clear and rigorous mathematical criteria for system effectiveness. Although the investigation was in part motivated by mistrust of the use of expected values (first moments) for such purposes, their use is evidently vindicated. A set of intuitively acceptable axioms which define the desiderata required of possible measures of effectiveness is established. It is then shown that these axioms are sufficient to determine a measure of effectiveness. Peripheral material on the theory of random sets relevant to effectiveness computations is presented in appendix form.

I.   INTRODUCTION

With the frequent and increasing need for systematic and reliable examination of large classes of system possibilities for determination of the more promising co   ırations, has come a correlative need for a systematic, reliable, and uniform method for rating their performance potential.   It is the objective of this study to produce at least the beginning of such a method. While no claim for completeness or finality is in order, it is clear that some significant progress toward this end is made in the formulations and results of this study.

The central problem which is broached in the study is in part semantic and in part mathematical.   It is in essence the question of what is meant by performance potential and how can clear mathematical expression be imposed upon the meaning, if such exists.   The approach is to establish several properties which a measure of performance potential, or effectiveness as it will be termed later, would seem to need.   The determination of need is intuitive, and therefore somewhat arbitrary; but it is only the results of such a determination which are incorporated into the analysis in the form of assumptions solely of a mathematical nature.   These assumptions appear as requirements which it is desired that any measure of performance potential satisfy.   As requirements, they intuitively appear as quite necessary properties of anything that could reasonably be conceived as a measure of performance potential of any significant generality.   It fortunately happens that these

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assumptions or requirements are sufficient to determine a quantitative measure of performance potential which is unique up to a constant multiplier, usually chosen equal to 1. It is this quantitative measure which is termed 'effectiveness'.

In the course of conducting this investigation a number of peripheral items have been explored, usually with inconclusive results and with no results of immediate applicability to the central problem. One of these items, however, is important enough to the development of effectiveness applications and further theory that, although it is only incompletely incorporated into the general abstract formulation, it is nevertheless presented in some detail. This item pertains to the theory of random sets and contains some very useful generalizations of existing theory. This material is treated independently of the rest and is mainly developed in Appendix I.

Similarly, an extension of the basic notion of effectiveness to include a 'conditional' effectiveness, which is not complete enough to fit smoothly into the general development, is considered in Appendix II. This material too, when properly developed, should provide a broader basis for an effectiveness analysis and allow a component analysis of larger systems well adapted to disclosing weak links.

II. DISCUSSION

Two of the major difficulties implicit in the development of a
satisfactory theory for rating performance potential are the ambiguities
usually associated with the objectives of goals of a system and the inherent
randomness of the operation of most systems.   The first of these difficulties
is ignored for the most part in this analysis, although it does not appear
entirely outside the scope of effectiveness theory.   The analysis assumes that
the goals of the system in all situations are known and that it is also known
when these goals are or are not gained by the operating sys.em.   These
assumptions, or rather the details withheld by these assumptions, are brought
into the general analysis by initially assuming that the result of operating a
system for given objectives under given conditions can be represented by a real
valued non-negative random variable.   Later it is shown by example how, in
cases of practical significance, these random variables can be obtained from a
detailed statement of the goals, their random elements, and the random actions
and reactions of the system.   The second of these difficulties, and perhaps the
greater from the aspect of the typical analysis of a system, is the main
objective of the investigation.

The difficulty of inherent randomness in its full generality and
vagueness can be stated: given that the work that a system does, or the value
of the goals it obtains, is a random variable, what knowledge of this random
variable is necessary to determine how well the system is performing its

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assignment and how is this determination to be made, particularly, when the assignment itself must be considered random. A not too illuminating version of this difficulty is frequently seen as some conundrum similar to "Is it better to kill half the enemy all the time or all of the enemy half of the time?"

To clarify at least the basis for such a conundrum, consider simple problems which will alternately be considered from different viewpoints. Let there be a simple system designed to determine the presence or absence of (say) a particle in a given volume of space at each instant of time. Let the only property of the system that might interfere with such a determination be the time to failure of the system. For simplicity let the time to failure have an exponential distribution given by

$$\Pr\left\{ T* \leq t \right\} = 1 - e^{-t/T}$$

when $T*$ is the random time to failure and T is the mean time to failure. Let the time interval of interest be from 0 to T_o. Let the probability that a particle ever appears be p and let the time at which it appears have a uniform distribution from 0 to T_o when it does appear.

Now that the situation has been defined, consider what the objective of such a simple system might be. It might be that the goal of the system is to provide certain knowledge of whether the particle did or did not appear. It might be to provide knowledge only when the particle does appear. It might be to determine whether the particle appeared at least with a probability (say) c. All of these objectives are somewhat similar and yet in important

ways dissimilar: all of them might be the objectives of the system under different circumstances, or under the same circumstances as seen by different people.

In the first case certain knowledge can be obtained if, and only if, either the particle appears and is detected or the particle does not appear and the system fails after T_o. Hence, in the first case the effectiveness might be the probability of this event, which is $pT (1 - e^{-T_o/T})/T_o + (1 - p) e^{-T_o/T}$. In the second case the required knowledge is provided if, and only if, the particle either appears and is detected or does not appear. Again the effectiveness might reasonably be identified with the probability that the required knowledge is supplied; the effectiveness is then $1 - p + pT (1 - e^{-T_o/T})/T_o$. The magnitude of the difference between these quantities is $(1 - p) (1 - e^{-T_o/T})$ so that not only are the formula different but also the results may be significantly different. In the third case the effectiveness is 0 or 1 as $e^{-T_o/T}$ is less than or not less than c, when that case is handled as the preceding ones. Thus, there is ample ground for confusion and, needless to say, when more than one particle is involved matters are worse.

As important and as extreme as this confusion is, it is not of direct consequence for the question of what knowledge of the random variables characterizing the work the system accomplishes and the work assigned is necessary for the determination of effectiveness, or what means are to be used in the determination. The present analysis assumes that a proper choice of random variables has been made, if such can be done.

The interpretation of the random variables designated 'work assigned' and 'work accomplished' is that the random variables considered as functions on their underlying probability space take as values quantities that would represent an actual assignment and an actual accomplishment if the random effects were suppressed. Thus 'work assigned' will be taken to mean that body of material or things which a system is desired to transform in some specified way, or some real valued function of that body which can be interpreted as a value or worth assignment. Similarly, 'work accomplished' will be taken to mean that portion of the body properly transformed or a real valued function of that portion.

Usually the work assigned and the work accomplished will be non-negative, real valued random variables. Occasionally, however, these random variables may degenerate into real numbers. Thus, there is a special case in which the effectiveness functional must be evaluated for real arguments. In this case, one can require with confidence, that the effectiveness increase in direct proportion to the accomplished work for a constant assigned work. Thus, if W designates the assigned work, S designates the accomplished work, ef(.,.) the effectiveness functional, and C(.) an unspecified positive decreasing function, then

ASSUMPTION I

The restriction of the effectiveness functional to the real numbers satisfies

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$$ef(S, W) = C(W)S \qquad (1)$$

when $W > 0$ and $0 \leq S \leq W$.

When the work assigned and the work accomplished are not necessarily
degenerate random variables, it is necessary to proceed differently. The
essence of the next requirement is balance: if the work assigned and the work
accomplished are both augmented in such a manner that the balance between
the two is unchanged, then the effectiveness is unchanged. In different words
the requirement might be that the effectiveness remain constant when the
assigned work varies, provided that the accomplished work varies
proportionately. In dealing with random quantities a better vantage can be
obtained by looking at 'proportionately' from a viewpoint different from the
usual.

Let f be a continuous real valued function of a real variable. Let f
be non-negative and increasing for an increasing non-negative argument; also
let $f(0) = 0$. Such a function is of the form $f(x) = mx$, $m > 0$ if, and only if,
equal increments in the argument produces equal increments in the values:
the relation of direct proportion is equivalent to that of equal increments in
the argument producing equal increments in the values. It is obvious that
direct proportion requires that equal increments produce equal increments.
When equal increments produce equal increments it follows that for each non-
negative h and all non-negative x and y the equation $f(x + h) - f(x) = f(y + h) - f(y)$
holds. Hence, as $f(0) = 0$ it follows that $f(x + h) = f(x) + f(h)$. Now if n is a

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positive integer, $f(nx) = f(\overline{n-1}x + x) = f(\overline{n-1}x) + f(x)$ so that $f(nx) = nf(x)$.

Similarly $f(x) = f(n\frac{x}{n}) = nf(\frac{x}{n})$ so that $f(\frac{x}{n}) = \frac{1}{n} f(x)$. Let m be a positive integer

so that $\frac{m}{n}$ is an arbitrary positive rational. Then, if $r = \frac{m}{n}$ it follows that

$f(rx) = f(m\frac{x}{n}) = mf(\frac{x}{n}) = \frac{m}{n}f(x) = rf(x)$. As f is continuous it follows that for any

real $a > 0$ the equation $f(ax) = af(x)$ holds. Therefore, with $a = x^{-1}$, it follows

that $f(x) = f(1)x$ as was to be shown.

Considering the balance requirement from the vantage of this equivalent

of direct proportion facilitates framing the requirement more unambiguously.

Increments in the work assigned can be interpreted as 'equal' in two distinct

cases; when all increments are the value of the same random variable so that

all increments are equal in the ordinary sense; and when all the increments

are identically and independently distributed and are 'equal' in an extended

sense. The equality of increments of the work assigned can be similarly

defined. Thus, the balance requirement can be stated: when equal increments

of work assigned produce equal increments of the work accomplished the

effectiveness remains constant, or

ASSUMPTION II

If the families of non-negative random variables S^*, $S^*_1, \ldots, S^*_N$ and

W^*, $W^*_1, \ldots, W^*_N$ respectively are identically and independently

distributed or are respectively the values of two random variables

(possibly dependent) then,

$$ef(S^*, W^*) = ef(S^*_1 + \ldots + S^*_N, \ W^*_1 + \ldots + W^*_N) \tag{2}$$

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The intended interpretation is that $S*$, $S*_1, \ldots, S*_N$ respectively represents the work accomplished on the respective assignments $W*$, $W*_1, \ldots, W*_N$. Unfortunately, this assumption appears more complex than it is, and further elaboration must be deferred to the next section.

The next requirement governs those cases in which it is necessary to consider limiting operations on the effectiveness functional. Since it is clearly necessary that systems with identically distributed work assigned and work accomplished have the same effectiveness, it seems necessary to require that, as the assignments and accomplishments of a given system become probabilistically indistinguishable from a limiting assignment and accomplishment, the effectiveness too must become indistinguishable from that of the limiting assignment and accomplishment. Therefore,

ASSUMPTION III

If $S*_n$ converges to $S*$ almost everywhere and $W*_n$ converges to $W*$ almost everywhere, then

$$ef(S*, W*) = \lim_{n, m \to \infty} ef(S*_n, W*_m) \qquad (3)$$

Here convergence almost everywhere is taken as in Halmos I: if $X*_n$ converges almost everywhere to $X*$, then the probability that $\lim_{n \to \infty} X*_n \neq X*$ is zero so that as n increases $X*_n$ becomes indistinguishable from the limiting random variable $X*$.

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Usually the work "system" is used ambiguously. On one hand it is used to designate particular hardware items and a more or less specific operating procedure, and on the other an aggregate of these items from the aspect of their common structure, design, and use. Here "system" will be taken to mean the latter. "System instance" will be reserved for the individual hardware items belonging to the system aggregate. More formally, a system is any well defined collection of system instances for which there exists an instruction for allocating the work assigned to the system to its instances and an instruction governing the operation of each of the instances on the assigned work.

From the vantage of these definitions it is so natural to identify the work done by the system with the totality of work done by its instances that it is clear that such an assumption is quite reasonable. Let G be a set of goals and $G_0 \subset G$ be those assigned a given system. Then the i-th instance of the given system will accomplish some random set $A^*_i \subset G_0$ of goals.[1] Let V be a function which assigns to each set of goals the real number representing the work or value associated with the set of the goals, then

ASSUMPTION IV

If the goals accomplished by the i-th instance of a system with N instances are A^*_i (i = 1,..., N) and if the goals accomplished by the system are A* then

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[1] For a discussion of random sets see Appendix I

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$$A* = A*_1 \cup A*_2 \cup \ldots \cup A*_N; \tag{4}$$

and if the value function V is a measure on a σ -ring of subsets of

G_o then

$$V(A*) = V(A*_1) + \ldots + V(A*_N) \tag{5}$$

so that the work done by the system is the sum of the work done by

the instances.

Now if a system of N identical instances is considered; and if each

instance is assigned the amount W*, and if each instance accomplishes S*,

then by (5) and (2)

$$ef(S*, W*) = ef(NS*, N\dot{W}*). \tag{6}$$

Since S* can be replaced by $\dfrac{S*}{N}$ and W* by $\dfrac{W*}{N}$ it follows that

$$ef(S*, W*) = ef(\frac{S*}{N}, \frac{W*}{N});$$

and if S* is replaced by MS* and W* by MW* for positive integer M that

$$ef(MS*, MW*) = ef(rS*, rW*)$$

for $r = \dfrac{M}{N}$; and therefore by (6) that

$$ef(S*, W*) = ef(rS*, rW*) \tag{7}$$

in which, of course, r can be any positive rational. If a is any real number,

there is a sequence of rationals a_n such that $a = \lim_{n \to \infty} a_n$. By (7)

$$ef(S*, W*) = ef(a_n S*, a_n W*)$$

so that

$$ef(S*, W*) = \lim_{n \to \infty} ef(a_n S*, a_n W*).$$

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As $a_n S*$ and $a_n W*$ converge almost everywhere to $aS*$ and $aW*$ respectively, it follows that

$$ef(S*, W*) = ef(aS*, aW*) \tag{8}$$

or the effectiveness is invariant with respect to dimension change.

An instance of any system can be conceived as belonging to another system composed of replicas of the given instance, all of which are probabilistically the same. If the original instance in its original system is assigned the work $W*$ and accomplishes the work $S*$, then each replica in the system of replicas can be conceived to have the assigned work $W*_i$ and accomplish $S*_i$ (for the i-th replica). The $S*_i$ and the $W*_i$ can also be conceived as families of independent random variables. In such a case it appears quite reasonable that the effectiveness of the system of replicas should be independent of the number (greater than zero) of replicas involved. This independence is in fact guaranteed by assumption II; so that as assumption IV allows the total assigned work to be represented as the sum

$$W*_T = W*_1 + \ldots + W*_N \tag{9}$$

and the total accomplished work by the sum

$$S*_T = S*_1 + \ldots + S*_N \tag{10}$$

it follows by assumption II that

$$ef(S*, W*) = ef(S*_T, W*_T). \tag{11}$$

Since equation (8) holds for any permissible $S*$ and $W*$ it also holds for $S*_T/N$ and $W*_T/N$. Hence

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$$\text{ef}(S^*_T, W^*_T) = \text{ef}(NS^*_T/N, NW^*_T/N) = \text{ef}(S^*_T/N, W^*_T/N);$$

and therefore

$$\text{ef}(S^*, W^*) = \lim_{N \to \infty} \text{ef}(S^*_T/N, W^*_T/N). \tag{12}$$

It is now only necessary to evaluate the limit on the right of (12). By (9) and (10) it follows that

$$\frac{S^*_T}{N} = \frac{1}{N} \sum_1^N {}_i S^*_i \text{ and } \frac{W^*_T}{N} = \frac{1}{N} \sum_1^N {}_i W^*_i$$

Since the variances of the S^*_i and the W^*_i can be assumed finite, it follows by the strong law of large numbers that S^*_T/N and W^*_T/N respectively, converge almost everywhere to the expected values ES^* and EW^*. Therefore, by assumption III it follows that

$$\text{ef}(S^*, W^*) = \text{ef}(ES^*, EW^*), \tag{13}$$

an important result which, independently of assumption I, states that the effectiveness of any system is dependent only on the expected values of the work assigned and the work accomplished. In fact, if in (8) a is chosen equal to $1/EW^*$, then

$$\text{ef}(S^*, W^*) = \text{ef}(ES^*/EW^*, 1) \tag{14}$$

so that the effectiveness depends only on the ratio of the average work accomplished to the average work assigned, regardless of the distributions involved provided they have a finite variance.

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If in (8) S* and W* are replaced by S and W respectively, then by assumption I

$$ef(aS, aW) = C(aW)aS$$

so that if $a = 1/W$ then by (8)

$$ef(S, W) = C(1)S/W.$$

Thus if $C(1)$ is taken to be 1 then by (13)

$$ef(S*, W*) = ES*/EW*. \tag{15}$$

Therefore, if the work assigned and the work accomplished have finite variances the effectiveness of the system is the ratio of the average work accomplished to the average work assigned.

As a simple illustration of this result consider a case in which the system is to accomplish N similar objectives, which are such that the system has a probability p of accomplishing any one. If no other factors are involved and the value associated with each such objective is unity, then the work assigned is simply N and the work accomplished has a binomial distribution with mean Np; so that by (15) the effectiveness is simply p. If the number of objectives were itself random, say with a binomial distribution, having rN as mean, then the work accomplished has a binomial distribution with mean Nrp. Thus, as might be expected, by (15) the effectiveness is again p. In the event that there is a random mixture of goals such that a goal of the i-th type occurs with the relative frequency r_i and has an associated value V_i and the system has a probability p_i of accomplishing a goal of the i-th type, then

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the effectiveness is simply

$$\left(\sum_1^N i \ V_i \ p_i \ r_i \right) \Big/ \left(\sum_1^N i \ V_i \ r_i \right) \tag{16}$$

when, now, N is the number of types involved. In the event that the goals have unit value but are such that the system has a probability 1-S of being rendered inoperative by an engagement with one, then the number of engagements survived has a geometric distribution. If S is the survival probability then pS^{n-1} is the probability that the n-th goal is accomplished. Hence, if N goals are assigned, the effectiveness is

$$\frac{p}{N} \cdot \frac{1-S^N}{1-S}.$$

If S goes to 1 then the effectiveness goes to p as before. However, if as before a random number of goals is assigned, having a binomial distribution with mean Nr, the effectiveness is

$$\frac{p}{rN} \cdot \frac{1-(1 - r + rS)^N}{1-S};$$

and is no longer in either case, independent of the number of goals assigned.

A final and less trivial example can be derived from the illustration of Appendix I. There it is shown that a missile with a circular normal damage function specified by the standard deviation σ_D functioning at a point with a circular normal distribution specified by the standard deviation σ_B relative to a target with a total value V_o spread out with a circular normal density of standard deviation σ_T accomplishes the average work

$$\frac{V_o \, \sigma_D^2}{\sigma_D^2 + \sigma_B^2 + \sigma_T^2}.$$

Since the work assigned is V_o, the effectiveness is $\sigma_D^2 / (\sigma_D^2 + \sigma_B^2 + \sigma_T^2)$.
If instead of a particular target with the spread σ_T, the target is of a random
spread uniformly distributed from S_1 to S_2, then so long as the total value is
not dependent on spread the effectiveness is by (15)

$$\frac{\sigma_D^2}{(S_2 - S_1)\sqrt{\sigma_D^2 + \sigma_B^2}} \tan^{-1}\left[\frac{(S_2 - S_1)\sqrt{\sigma_D^2 + \sigma_B^2}}{\sigma_D^2 + \sigma_B^2 + S_1 S_2}\right] \quad (17)$$

In the event that V_o is dependent on σ_T a more complex expression holds.

APPENDIX I

THE EXPECTED VALUE OF THE RANDOM
MEASURE OF A RANDOM SET

The purpose of this appendix is to make mathematically precise the notion of a random set and a random measure on a random set and to develop a formula to compute its expected value. Finally, an illustrative example, with relevance to effectiveness computations is demonstrated.

In 1944 H. E. Robbins introduced the notion of a random set (Robbins I) and proved a theorem concerning the moments of its Lebesgue measure. This paper was followed by another (Robbins II) in which the theorem is applied to a problem of Bronowski and Neyman (Bronowski and Neyman I). For the purposes of effectiveness theory it is necessary to generalize these results of Robbins in order to allow non-uniform value distributions which themselves may be random.

Although with minor modifications the argument herein presented generalizes Robbins' results for all moments of a σ-finite measure of a random set, only the case of the first moment is considered; since only the first moment if relevant to an effectiveness computation.

In Robbins I a random set is defined by analogy with an example: X_n^* (n = 1,..., N) is a sequence of N independently and identically distributed random variables on a segment of the real line. About each X_n^* is centered an interval of unit length. These intervals and their unions are random sets.

In this appendix a measure-theoretic definition is introduced, the results are generalized to the σ -finite case, measures which are themselves random variables are considered, and the proofs are simplified considerably. Throughout, unless the contrary is stated, the measure-theoretic nomenclature is that of Halmos' Measure Theory (Halmos I).

Given are a probability space (Ω , $\overline{\Omega}$, p) and a σ - finite measure space (R, $\overline{R}$, V). Loosely the interpretation is that (Ω , $\overline{\Omega}$, p) carries the random element of a system's interaction with its goals in R and (R, $\overline{R}$, V) carries the value element in that if R_o $\in$ $\overline{R}$ is a set of goals accomplished by the system, then $V(R_o)$ is the value of the accomplishment. If, for example, R were a target represented by the Euclidian plane, then V might be the Lebesgue-Stieltjes measure induced by the personnel density function. (Ω , $\overline{\Omega}$, p) might carry the random impact point of a missile and its random lethal area. However, since the random interaction of the system with its goals is not necessarily independent of the goals, attention will be focussed on the product space (ΩXR, $\overline{\Omega}$X$\overline{R}$, p XV).

Let S be a measurable subset of Ω XR, that is S $\in$ $\overline{\Omega}$X$\overline{R}$. Each S determines a natural mapping of Ω into $\overline{R}$, the measurable subsets of R: let S_ω be an ω -section of S and for each $\omega \in \Omega$ pointwise define

$$S*(\omega) = S_\omega .\tag{I-1}$$

As all sections of a measurable set are measurable, S* maps Ω into $\overline{R}$. The mapping S* is the random set on R generated by S. As at least the countable

union and the relative complement of measurable subsets of Ω XR are measurable, the random sets on R form a σ-ring when pointwise definitions of union and relative complement are given.

As a consequence of 35 A Halmos I, $V(S_\omega)$ is a p-measurable function of ω, and consequently $V(S*)$ can be considered a random variable in the ordinary sense. Also, when S^r is an r-section of S, the same theorem yields

$$\int_\Omega V(S_\omega)dp(\omega) = \int_R p(S^r)dV(r) \qquad (I-2)$$

The left of (I-2) is familiar as the usual definition of the expected value of a random variable. Thus

$$E\left[V(S*)\right] = \int_R p(S^r)dV(r), \qquad (I-3)$$

when E is the expectancy functional. The set S^r is by definition $\left\{\omega : (\omega, r) \in S\right\}$. But as $(\omega, r) \in S$ if and only if $r \in S*(\omega)$ it can be identified with the event that r is in $S*$. Hence $p(S^r)$ is the probability that $r \in S*$. Accordingly if reference to the Ω space is dropped and 'Pr' is used to designate the probability of an event, (I-3) gives rise to the basic theorem for the expected value of the σ-finite measure of a random set:

$$E\left[V(S*)\right] = \int_R Pr(r \in S*) dV(r) \qquad (I-4)$$

Equation (I-4) generalizes Robbins I to the case of σ-finite V, and supplies a useful tool for the computation of effectiveness. Also from the definition of a

product measure it follows that

$$E\left[V(S*)\right] = (PXV)(S),\qquad\qquad\qquad (I-5)$$

or that the expected value of the V-measure of the random set is the product

measure of its generating set.

For some effectiveness computations it is necessary to consider value

measures which depend on conditions that can be specified only randomly.

These cases require a notion of random measure, and require the computation

of the expected value of the random measure of a random set. To fix the idea

of a random measure consider another probability space $(\Omega', \overline{\Omega}', p')$ which

is such that for almost all $\omega' \in \Omega'$ there is a totally finite value measure $V*_{\omega'}$

on $(R, \overline{R})$. Then (Halmos I, 36.3) for $R_o \in \overline{R}$ and $\emptyset(\omega') = V*_{\omega'}(R_o)$ the function

$\emptyset$ is p'-measurable and, consequently, suppressing the ω', $V*(R_o)$ is a random

variable in the ordinary sense. In this sense a random measure $V*$ is defined

for $(R, \overline{R})$. Now as $Pr(r \in S*)$ is a non-negative measurable function on R, it

follows that (Halmos I, 36.3)

$$f(\omega') = \int_{\Omega'} Pr(r \in S*)\ dV_{\omega'}(r),$$

the expected value of the measure of S* given that $V* = V_{\omega'}$, is measurable on

Ω' and that

$$\int_{\Omega'} f(\omega')dp'(\omega') = \int_R Pr(r \in S*)d\ (EV*)(r)\qquad\qquad (I-6)$$

when EV* is the expected measure as defined by

I-4

$$(EV*)(R_o) = \int_{\Omega'} V_{\omega'}(R_o)dp'(\omega').$$

Thus, as the left of (I-6) is by definition the expected value of $V_{\omega'}(S*)$, it follows that

$$E\left[V*(S*)\right] = \int_R Pr(r \in S*)d \,(EV*)(r). \qquad (I-7)$$

Equation (I-7) shows that insofar as effectiveness computations are concerned, a random measure can be treated in the same manner as a non-random measure by using its expected value. In this sense (I-7) is a further generalization of (I-4); however, in the derivation of (I-7) the assumption of σ-finite measure is replaced by one of a totally finite measure. But, as value measures usually can be considered totally finite no important restrictions are thereby imposed on an effectiveness computation.

Now it is desirable to show that the product space representation of the requirements for an effectiveness analysis is essentially complete. This is to be accomplished by generalizing (I-5) as (I-6) is generalized. Thus, not only allow the value measure (again totally finite) to depend on conditions that can be specified only randomly but also allow p to be so dependent. Accordingly, for each $\omega' \in \Omega'$ let $p_{\omega'}XV$ be a totally finite measure on (ΩXR, $\bar{\Omega} X\bar{R}$) Then for $S \in \bar{\Omega} X\bar{R}$ and $\emptyset(\omega') = (p_{\omega'} XV_{\omega'})^*(S)$ the function $\emptyset$ is p'-measurable as before By (I-5), $\emptyset(\omega')$ is the conditional expectation of the random set generated by S. Since $\emptyset$ is measurable and finite it is integrable. Therefore,

I-5

$$E\left[V*(S*)\right] = \int_{\Omega'} \phi(\omega')dp(\omega') \tag{I-8}$$

and as before define an expected measure λ by

$$\lambda(S) = \int_{\Omega'} \phi(\omega')dp(\omega')$$

so that by (I-8), in analogy with (I-5), it follows that

$$E\left[V*(S*)\right] = \lambda(S), \tag{I-9}$$

and therefore that ($\Omega\,XR,\ \bar{\Omega}\,X\bar{R},\ \lambda$) is sufficient to determine the effectiveness.

Continuing in this vein it, of course, follows that if only the p-measure on (Ω, $\bar{\Omega}$) is randomized, or if both measures are independently randomized an equation of the form (I-3) can be shown to hold with p or V replaced by Ep* or EV* as the case may require. Thus, in many cases the effectiveness is representable by (I-4) through suitable choice of p and V. In all cases of practical significance it is representable by (I-9) and therefore the effectiveness problem is essentially specified by ($\Omega\,XR,\ \bar{\Omega}\,X\bar{R},\ \lambda$) in which ($\Omega$, $\bar{\Omega}$) is an event space (R, $\bar{R}$) a goal space and λ a totally finite measure on their product space.

It is to be emphasized that the above characterization is not to be considered final. On a general and abstract level it appears suitable and adequate. Such, however, is not to state that the characterization is adequate for unforeseen complex problems.

As an illustrative example of these techniques consider the following:

APPENDIX II

CONDITIONAL EFFECTIVENESS

The notion of conditional effectiveness is an outgrowth of the application of the preceding development of effectiveness to the determination of the degree to which a component of a system contributes to the effectiveness of the system. The aim is a quantitative measure of how well a particular component, or a characteristic of a particular component, functions in relation to the rest of the system. Such a measure should be of value in the rapid assessment of system weak-links, and should serve as an indicator of which elements of a system could be improved most profitably.

To establish such a measure in the context of the previous developments it is first necessary to determine the work assigned. Since a relative measure is required it seems natural to define the work assigned as the work that would be accomplished by the system were the component in question functioning perfectly. Such a definition seems to properly account for the limitations imposed on the performance of the component in question by possible inadequacies and mismatching of other system components. With the work assigned so defined the work accomplished can be identified with the work accomplished by the entire system.

As an example consider a system composed of N components each with a probability $p_i (i = 1, \ldots, N)$ of proper performance. Then the conditional effectiveness of the k-th component is simply

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